Solving a System of Linear Equations in Three Variables

Steps for Solving

Step 1: Pick two of the equations in your system and use elimination to get rid of one of the variables.
Step 2: Pick a different two equations and eliminate the same variable.
Step 3: The results from steps one and two will each be an equation in two variables. Use either the elimination or substitution method to solve for both variables.
Step 4: Substitute the values found in step 3 into any one of the original three equations to find the value of the third variable.

Example

Let's look at the following system:

$$x + 2y - z = 4$$
$$3x - y + z = 5$$
$$2x + 3y + 2z = 7$$

It's important to do the work neatly and systematically in these types of problems. There are many steps and it's easy to make small errors. Labeling the equations before starting the problem can help.

A $x + 2y - z = 4$

B $3x - y + z = 5$

C $2x + 3y + 2z = 7$

Now, we want to pick two of the equations and eliminate one of the variables from those two equations. It doesn't matter which two, and it doesn't matter which variable. In some problems, some variables are easier than others. In this case, it looks like z will be an easy variable to eliminate from equations A and B because the coefficients of z are opposites in those two equations.

To eliminate z from equations A and B, we just need to add the equations together:

A $x + 2y - z = 4$

B $\underline{3x - y + z = 5}$

$4x + y = 9$ **D**

We'll call this new equation in y and x equation D.

Now, we need to pick a different two equations, either A and C or B and C, and eliminate the same variable, z, from those two. Since the coefficients of z in equations A and C have opposite signs, these would be the easiest two to work with. We will have to multiply equation A by 2 in order to eliminate z.

$2 \cdot$ **A** $= (2)x + (2)2y - (2)z = (2)4 \rightarrow 2x + 4y - 2z = 8$

C $\underline{2x + 3y + 2z = 7}$

$4x + 7y = 15$ **E**

We'll call this new equation equation E.

If we just look at equations D and E, we have a system of two equations in two variables, which we already know how to solve. We can use either the substitution or elimination method.

Since the coefficient of x is 4 in both equations, we can eliminate x by multiplying one of the equations by -1.

$-1 \cdot$ **D** $= (-1)4x + (-1)y = (-1)9 \rightarrow -4x - y = -9$

E $\underline{4x + 7y = 15}$

$6y = 6$

$y = 1$

(Note: We could also use the substitution method to solve for x or y.)

Now we have one part of our answer. We can substitute the value of y back into either equation D or E to find x.

D $4x + y = 9 \rightarrow 4x + 1 = 9$

$\underline{\quad - 1 = -1}$

$4x = 8$

$x = 2$

Now we have the second part of our answer. Once we have solved for two of the variables, we can substitute those values back into any one of the original three equations to solve for the last variable.

B $3x - y + z = 5 \rightarrow 3(2) - (1) + z = 5$

$6 - 1 + z = 5$

$5 + z = 5$

$\underline{-5 \quad -5}$

$z = 0$

So, the solution for this system is (2, 1, 0). Since there is one unique solution, we say that the system is consistent and independent.

Types of Solutions

A system of three equations may have one unique solution, infinitely many solutions, or no solution.

- If there is one solution, we say that the system is consistent and independent.
- If there are infinitely many solutions, we say that the system is consistent and dependent.
- If there is no solution, we say that the system is inconsistent.

How do we know what type of system we have? Consistent independent systems are easy, since we get an answer. But what do consistent dependent and inconsistent systems look like?

Let's consider the following example:

A $3x + 2y + z = 3$

B $x - 3y + z = 4$

C $-6x - 4y - 2z = 1$

It looks like we can easily eliminate x from equations A and C by multiplying equation A by 2 and adding them together.

$$\begin{array}{lr} 2 \cdot \mathbf{A} \rightarrow & 6x + 4y + 2z = 6 \\ \mathbf{C} & \underline{-6x - 4y - 2z = 1} \\ & 0 = 7 \end{array}$$

All of our variables have cancelled out, and we are left with an untrue statement, or contradiction. This tells us that there is no solution, so the system is inconsistent.

Sometimes we have to do a little more work before we realize that the system does not have a unique solution. Let's look at the next example.

A $x + 2y - 7z = -4$

B $2x + y + z = 13$

C $3x + 9y - 36z = -33$

If we multiply equation A by -2 and add it to equation B, we can eliminate x.

$$\begin{array}{lrl} -2 \cdot \mathbf{A} \rightarrow & -2x - 4y + 14z = 8 & \\ \mathbf{B} & \underline{2x + y + z = 13} & \\ & -3y + 15z = 21 & \mathbf{D} \end{array}$$

If we multiply equation A by -3 and add it to equation C, we can eliminate x.

-3 • **A** →	$-3x - 6y + 21z = 12$	
C	$\underline{3x + 9y - 36z = -33}$	
	$3y - 15z = -21$	**E**

Equations D and E are the opposites. If we add them together, everything is going to cancel out and we will be left with 0 = 0, which is a true statement, or an identity. When this happens, we have a consistent dependent system.

D	$-3y + 15z = 21$
E	$\underline{3y - 15z = -21}$
	$0 = 0$

So, how do we write the solution? In some math classes, it is okay to just write that the system is consistent dependent, but in others it is not. If not, we can write the system this way.

Let $z = a$.

(It doesn't matter which of the variables we pick, or what new variable we choose to set them equal to.)

Now, we substitute a for z in either equation D or E and solve for y.

$3y - 15a = -21 \rightarrow \quad y = 5a - 7$

Now we have z and y defined in terms of a. To get x in terms of a, we substitute $z = a$ and $y = 5a - 7$ into any one of the three original equations.

$x + 2(5a - 7) - 7a = -4$

Solving, we get $x = -3a + 10$.

Our solution is $(-3a + 10, \ 5a - 7, \ a)$.