Complex Numbers and Powers of $i$

**The Number $i$** - $i$ is the unique number for which $i = \sqrt{-1}$ and $i^2 = -1$.

**Imaginary Number** – any number that can be written in the form $a + bi$, where $a$ and $b$ are real numbers and $b \neq 0$.

**Complex Number** – any number that can be written in the form $a + bi$, where $a$ and $b$ are real numbers. (Note: $a$ and $b$ both can be 0.) The union of the set of all imaginary numbers and the set of all real numbers is the set of complex numbers.

**Addition / Subtraction** - Combine like terms (i.e. the real parts with real parts and the imaginary parts with imaginary parts).

*Example* - $(2 - 3i) - (4 - 6i) = 2 - 3i - 4 + 6i = -2 + 3i$

**Multiplication** - When multiplying square roots of negative real numbers, begin by expressing them in terms of $i$.

*Example* - $\sqrt{-4} \cdot \sqrt{-8} = \sqrt{-1} \cdot \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{8}$

$= i \cdot 2 \cdot i \cdot 2\sqrt{2}$

$= i^2 \cdot 4\sqrt{2}$

$= (-1) \cdot 4\sqrt{2}$

$= -4\sqrt{2}$

*Note*: The answer is not $+4\sqrt{2}$, which could be calculated erroneously if the radicands were simply multiplied as $\sqrt{-4} \cdot \sqrt{-8} \neq \sqrt{(-4)(-8)} \neq \sqrt{32}$
**Multiplication (Cont’d)** – When multiplying two complex numbers, begin by FOILing them together and then simplify.

*Example* - \((2 + 3i) \cdot (8 - 7i)\)

\[
\begin{align*}
16 - 14i + 24i - 21i^2 &= 16 + 10i - 21i^2 \\
&= 16 + 10i - 21(-1) \\
&= 16 + 10i + 21 \\
&= 37 + 10i
\end{align*}
\]

**Division** – When dividing by a complex number, multiply the top and bottom by the complex conjugate of the denominator. Then FOIL the top and the bottom and simplify. The answer should be written in standard form \((a + bi.)\)

*Example* - \(\frac{2+3i}{1-5i} \cdot \frac{(1+5i)}{(1+5i)}\) (Multiply by complex conjugate)

\[
\begin{align*}
\frac{2+10i+3i+15i^2}{1+5i-5i-25i^2} &= \frac{2+13i+15(-1)}{1-25(-1)} \\
&= \frac{2+13i-15}{1+25} \\
&= \frac{-1+i}{2} = -\frac{1}{2} + \frac{1}{2}i
\end{align*}
\]

*Example* - \(\frac{14}{i} \cdot \frac{-i}{-i}\) (Multiply by complex conjugate)

\[
\begin{align*}
\frac{-14i}{-i^2} &= \frac{-14i}{-(-1)} \\
&= \frac{-14i}{1} = -14i
\end{align*}
\]
Powers of $i$ – Given a number, $i^n$, the number can be simplified by using the following chart.

<table>
<thead>
<tr>
<th>$i^n$</th>
<th>Is Equivalent to…</th>
<th>Because…</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i^0$</td>
<td>1</td>
<td>a number raised to the 0 power is 1</td>
</tr>
<tr>
<td>$i^1$</td>
<td>$i$</td>
<td>a number raised to the 1 power is that same number</td>
</tr>
<tr>
<td>$i^2$</td>
<td>$(-1)$</td>
<td>$i^2 = -1$ (definition of $i$)</td>
</tr>
<tr>
<td>$i^3$</td>
<td>$-i$</td>
<td>$i^3 = i^2 \cdot i = (-1) \cdot i = -i$</td>
</tr>
<tr>
<td>$i^4$</td>
<td>1</td>
<td>$i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1$</td>
</tr>
<tr>
<td>$i^5$</td>
<td>$i$</td>
<td>$i^5 = i^4 \cdot i = (1) \cdot i = i$</td>
</tr>
</tbody>
</table>

Because the powers of $i$ will cycle through $1, i, -1, and -i$, this repeating pattern of four terms can be used to simplify $i^n$.

**Example** - Simplify $i^{25}$

Step 1 - Divide 25 (the power) by 4.

$$\frac{25}{4} = \text{quotient of 6 with a remainder of 1}$$

Step 2 - Note the quotient (i.e. 6) and the remainder (i.e. 1).

Step 3 - Rewrite the problem.

$$i^{25} = (i^4)^{\text{quotient}} \cdot i^{\text{remainder}} = (i^4)^6 \cdot i^1$$

Step 4 - Simplify by recalling that $i^4 = 1$

$$(i^4)^6 \cdot i^1 = (1)^6 \cdot i^1 = 1 \cdot i = i$$

*Note:* Because the powers of $i$ cycle through $1, i, -1, and i$, these types of problems can always be simplified by noting what the remainder is in step 2 above. In fact, the problem can be re-written as...

$$i^n = i^{\text{remainder}} \quad \text{(Divide n by 4 and determine the remainder).}$$

*The remainder will always be either 0, 1, 2, or 3.*

**Example** - Simplify $i^{59}$

$$i^{59} = i^3 \quad \text{(because} \quad \frac{59}{4} \text{has a remainder of 3.)}$$

So, $$i^{59} = i^3 = -i$$
Imaginary and Complex Numbers Practice

Simplify:

1) \((4 + 2i) + (-3 - 5i)\)
2) \((-3 + 4i) - (5 + 2i)\)
3) \((-8 - 7i) - (5 - 4i)\)
4) \((3 - 2i)(5 + 4i)\)
5) \((3 - 4i)^2\)
6) \((3 - 2i)(5 + 4i) - (3 - 4i)^2\)

7) Write \(\frac{3 + 7i}{5 - 3i}\) in standard form
8) Simplify \(i^{925}\)
9) Simplify \(i^{460}\)
10) Write \(\frac{1 - 4i}{5 + 2i}\) in standard form
11) \(\sqrt{-16}\)
12) \(\sqrt{-8}\)
13) \(\sqrt{-6}\) \(\sqrt{-6}\)
14) \(4 + \sqrt{-25}\)
15) \(\frac{6 - \sqrt{-8}}{-2}\)

Answers:

1) \(1 - 3i\)  (2) \(-8 + 2i\)  (3) \(-13 - 3i\)  (4) \(23 + 2i\)
5) \(-7 - 24i\)  (6) \(30 + 26i\)  (7) \(-\frac{3}{17} + \frac{22}{17}i\)  (8) \(i\)
8) \(1\)  (10) \(-\frac{2}{29} - \frac{22}{29}i\)  (11) \(4i\)  (12) \(2\sqrt{2}i\)
13) \(-6\)  (14) \(4 + 5i\)  (15) \(-3 + \sqrt{2}i\)