Linear Asymptotes and Holes

Graphs of Rational Functions can contain linear asymptotes. These asymptotes can be Vertical, Horizontal, or Slant (also called Oblique). Graphs may have more than one type of asymptote. Given a Rational Function \( f(x) \), the steps below outline how to find the asymptote(s).

\[
Rational Function = f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}
\]

**Vertical Asymptotes (VA)** – The line \( x = c \) is a Vertical Asymptote of the graph of a rational function \( f(x) \) when \( f(x) \to \pm \infty \), as \( x \to c \) from the right or the left. They are graphed as dashed vertical lines.

The graph of \( f(x) \) has Vertical Asymptotes at the real zeros of \( D(x) \). Find the VA’s by setting the denominator of the simplified function equal to “0” and solving the resulting equation. Note: Verify that the real zeros of \( D(x) \) are not actually “holes” first by factoring \( N(x) \) and \( D(x) \) and simplifying the function if possible. (See “Holes*” at bottom of second page.)

**Horizontal Asymptotes (HA)** – The line \( y = c \) is a Horizontal Asymptote of the graph of a rational function \( f(x) \) when \( f(x) \to c \), as \( x \to \pm \infty \). They are graphed as dashed horizontal lines.

The graph of \( f(x) \) either has 1 Horizontal Asymptote or no HA which is determined by comparing the degree \( (n) \) of the Numerator \( N(x) \) with the degree \( (m) \) of the Denominator \( D(x) \).

There are 3 cases to consider.

<table>
<thead>
<tr>
<th>Case 1:</th>
<th>Case 2:</th>
<th>Case 3:</th>
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<tbody>
<tr>
<td>If ( n &lt; m )</td>
<td>If ( n = m )</td>
<td>If ( n &gt; m )</td>
</tr>
<tr>
<td>HA: ( y = 0 )</td>
<td>HA: ( y = \frac{a_n}{b_m} )</td>
<td>HA: None</td>
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But there could be a Slant Asymptote!

**Examples:** Find all Vertical and Horizontal Asymptotes of the graphs of the Rational Functions.

a) \( f(x) = \frac{x - 2}{x^2 - 9} = \frac{x - 2}{(x+3)(x-3)} \)

Set \( (x+3)(x-3) = 0 \)

So \( x = -3, 3 \)

VA: \( x = -3, x = 3 \)

HA: \( n < m \), so \( y = 0 \)

b) \( f(x) = \frac{3x^2 - 12}{x^2 + 2x - 3} = \frac{3(x+2)(x-2)}{(x+3)(x-1)} \)

Set \( (x+3)(x-1) = 0 \)

So \( x = -3, 1 \)

VA: \( x = -3, x = 1 \)

HA: \( n = m \), so \( y = \frac{a_m}{b_m} = \frac{3}{1} = 3 \)
Slant (or Oblique) Asymptotes (SA) – The line \( y = mx + b \) is a Slant Asymptote of the graph of a rational function if \( \lim_{{x \to \pm \infty}} f(x) = mx + b \). They are graphed as dashed lines.

If the degree of the numerator \((n)\) is exactly 1 more than the degree of the denominator \((m)\), then there could be a Slant Asymptote.

To find the Slant Asymptote:

1. Factor the numerator and denominator of \( f(x) \).
2. Identify and “reduce” any holes*. (See next section.)
3. Multiply the numerator and denominator back in to polynomials if necessary and then divide the remaining numerator by the remaining denominator (i.e. use long division).
4. Set the resulting quotient equal to \( y \). Ignore the remainder.
5. The resulting equation, \( y = mx + b \), is the Slant Asymptote.

Examples: Find any VA’s and Slant Asymptotes of the graphs of the Rational Functions.

a) \( f(x) = \frac{x^2-3x-4}{x-2} = \frac{(x-4)(x+1)}{x-2} \)

\begin{align*}
\text{Quotient} & \quad x - 1 \\
\text{Long Division:} & \quad x - 2 \sqrt{x^2 - 3x - 4} \\
& \quad x^2 - 2x \\
& \quad -x - 4
\end{align*}

\( \text{SA: } y = x - 1 \)

HA: None

VA: \( x = 2 \)

b) \( f(x) = \frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{x+3} \)

\( \text{So, } f(x) = x - 2 \) with a hole at \( x = -3 \)

\( \text{SA: } \) None \( \text{ (function no longer has a denominator) } \)

Holes* - Sometimes, graphs of Rational Functions can contain a “Hole(s)”. This occurs when a common (real) factor shows up in the numerator and denominator. This value of \( x \) is still a domain restriction, but it is represented as a “Hole” in the graph of \( f(x) \) vs. as a Vertical Asymptote.

Example - Find any Horizontal, Vertical, or Slant Asymptotes of \( f(x) \). Also identify any Holes.

\[ f(x) = \frac{2x^3-x^2-2x+1}{x^2+3x+2} = \frac{(x+1)(x-1)(2x-1)}{(x+1)(x+2)} = \frac{(x-1)(2x-1)}{(x+2)} = \frac{2x^2-3x+1}{x+2} \quad \text{\( n > m \) by exactly 1} \]

Holes: \( @ x = -1 \)

VA: \( x = -2 \)

HA: None \( (n > m) \)

SA: \( y = 2x - 7 \)