Finding Equations of Polynomial Functions with Given Zeros

Polynomials are functions of general form \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \) \((n \in \text{ whole #'s})\)

Polynomials can also be written in factored form \( P(x) = a(x - z_1)(x - z_2) \cdots (x - z_i) \) \((a \in \mathbb{R})\)

Given a list of “zeros”, it is possible to find a polynomial function that has these specific zeros. In fact, there are multiple polynomials that will work. In order to determine an exact polynomial, the “zeros” and a point on the polynomial must be provided.

Examples: Practice finding polynomial equations in general form with the given zeros.

**Find an* equation of a polynomial** with the following two zeros: \( x = -2, x = 4 \)

Denote the given zeros as \( z_1 \) and \( z_2 \)

**Step 1:** Start with the factored form of a polynomial.

\[
P(x) = a(x - z_1)(x - z_2)
\]

**Step 2:** Insert the given zeros and simplify.

\[
P(x) = a(x - (-2))(x - 4)
\]

\[
P(x) = a(x + 2)(x - 4)
\]

**Step 3:** Multiply the factored terms together.

\[
P(x) = a(x^2 - 2x - 8)
\]

**Step 4:** The answer can be left with the generic “\( a \)”, or a value for “\( a \)” can be chosen, inserted, and distributed.

i.e. if \( a = 1 \), then \( P(x) = x^2 - 2x - 8 \)

i.e. if \( a = -2 \), then \( P(x) = -2x^2 + 4x + 16 \)

*Each different choice for “\( a \)” will result in a distinct polynomial. Thus, there are an infinite number of polynomials with the two zeros \( x = -2 \) and \( x = 4 \).

**Find the equation of a polynomial** with the following zeroes: \( x = 0, -\sqrt{2}, \sqrt{2} \) that goes through the point \((-2, 1)\).

Denote the given zeros as \( z_1, z_2 \) and \( z_3 \)

**Step 1:** Start with the factored form of a polynomial.

\[
P(x) = a(x - z_1)(x - z_2)(x - z_3)
\]

**Step 2:** Insert the given zeros and simplify.

\[
P(x) = a(x - 0)(x - (-\sqrt{2}))(x - \sqrt{2})
\]

\[
P(x) = ax(x + \sqrt{2})(x - \sqrt{2})
\]

**Step 3:** Multiply the factored terms together

\[
P(x) = a(x^3 - 2x)
\]

**Step 4:** Insert the given point \((-2, 1)\) to solve for “\( a \)”.

\[
1 = a[(−2)^3 − 2(−2)]
\]

\[
1 = a[−8 + 4]
\]

\[
1 = −4a
\]

\[
a = −\frac{1}{4}
\]

**Step 5:** Insert the value for “\( a \)” into the polynomial, distribute, and re-write the function.

\[
P(x) = −\frac{1}{4}(x^3 − 2x) = −\frac{1}{4}x^3 + \frac{1}{2}x
\]
**Polynomials** can have zeros with *multiplicities greater than 1*. This is easier to see if the Polynomial is written in **factored form**.

\[ P(x) = a(x - z_1)^m(x - z_2)^n ... (x - z_i)^p \]

**Multiplicity** - The number of times a “zero” is repeated in a polynomial. The multiplicity of each zero is inserted as an exponent of the factor associated with the zero. If the multiplicity is not given for a zero, it is assumed to be 1.

**Examples:** Practice finding polynomial equations with the given zeros and multiplicities.

**Find an equation of a polynomial** with the given zeroes and associated multiplicities. *Leave the answer in factored form.*

<table>
<thead>
<tr>
<th>Zeros</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = 1)</td>
<td>2</td>
</tr>
<tr>
<td>(x = -2)</td>
<td>3</td>
</tr>
<tr>
<td>(x = 3)</td>
<td>1</td>
</tr>
</tbody>
</table>

**Step 1:** Write the factored form of the Polynomial.

\[ P(x) = a(x - 1)^2(x - 2)^3(x - 3)^1 \]

**Step 2:** Insert the given zeros and their corresponding multiplicities.

\[ P(x) = a(x - 0)^3(x - 1)^2(x - i)^1(x + i) \]

**Step 3:** Simplify any algebra if necessary. The answer can be left with the generic “\(a\)”, or a specific value for “\(a\)” can be chosen and inserted if requested.

\[ P(x) = ax^7 + 2ax^6 + 2ax^5 + 2ax^4 + ax^3 \]

**Find an equation of a polynomial** with the given zeros and associated multiplicities. *Expand the answer into general form.*

**Step 1:** Write the factored form of the Polynomial.

\[ P(x) = a(x - z_1)^m(x - z_2)^n ... (x - z_i)^p \]

**Step 2:** Insert the given zeros and their corresponding multiplicities.

**Step 3:** Simplify any algebra if necessary.

**Step 4:** Multiply the factored terms together. Recall that \(i^2 = -1\)! Note the generic “\(a\)” can be used and distributed, or a specific value for “\(a\)” can be chosen and inserted if requested.
Practice Problems: Try these problems on your own!

Find an equation of a Polynomial with the given zeros.

1. Zeros: $x = -1, -3$  
   \textit{Answer: } $f(x) = a(x^2 + 4x + 3)$

2. Zeros: $x = -2, 2, \sqrt{3}, -\sqrt{3}$  
   \textit{Answer: } $f(x) = a(x^4 - 7x^2 + 12)$

3. Zeros: $x = -4i, 4i$  
   \textit{Answer: } $f(x) = a(x^2 + 16)$

Find the equation of a Polynomial given the following zeros and a point on the Polynomial.

4. Zeros: $x = 0, -4$  Point: $(-3, 6)$  
   \textit{Answer: } $f(x) = -2x^2 - 8x$

5. Zeros: $x = -2, 5$  Point: $(2, -3)$  
   \textit{Answer: } $f(x) = \frac{1}{4} x^2 - \frac{3}{4} x - \frac{5}{2}$

6. Zeros: $x = -4, -1, 1$  Point: $(2, 9)$  
   \textit{Answer: } $f(x) = \frac{1}{2} x^3 + 2x^2 - \frac{3}{2} x - 2$

Find an equation of a Polynomial given the following zeros with the listed multiplicities. In each example, set $a = 1$.

7. Zero: $x = 3$, \textit{Multiplicity} 2  
   \textit{Answer: } $f(x) = x^2 - 6x + 9$

8. Zero: $x = 0$, \textit{Multiplicity} 2  
   Zero: $x = -2$, \textit{Multiplicity} 1  
   \textit{Answer: } $f(x) = x^3 + 2x^2$

9. Zero: $x = -1$, \textit{Multiplicity} 1  
   Zero: $x = 1$, \textit{Multiplicity} 1  
   Zero: $x = -i$, \textit{Multiplicity} 1  
   Zero: $x = i$, \textit{Multiplicity} 1  
   \textit{Answer: } $f(x) = x^4 - 1$

10. Zero: $x = -3$, \textit{Multiplicity} 2  
    Zero: $x = 2$, \textit{Multiplicity} 1  
    \textit{Answer: } $f(x) = x^3 + 4x^2 - 3x - 18$