Rational Expressions

A quotient of two integers, \( \frac{a}{b} \), where \( b \neq 0 \), is called a **rational expression**.

Some examples of rational expressions are \( \frac{7x}{9} \), \( \frac{12}{x+4} \), \( \frac{3x+1}{2x-5} \), and \( \frac{x^2-10}{x^3-x^2+3} \). When \( x = -4 \), the denominator of the expression \( \frac{12}{x+4} \) becomes 0 and the expression is meaningless. Mathematicians state this fact by saying that the expression \( \frac{12}{x+4} \) is undefined when \( x = -4 \). One can see that the value \( x = \frac{5}{2} \), makes the expression \( \frac{3x+1}{2x-5} \) undefined. On the other hand, when any real number is substituted into the expression \( \frac{7x}{9} \), the answer is always a real number. There are no values for which this expression is undefined.

**EXAMPLE**

Determine the value or values of the variable for which the rational expression is defined.

a) \( \frac{x+3}{2x-5} \)

b) \( \frac{x+3}{x^2+6x-7} \)

**Solution**

a) Determine the value or values of \( x \) that make \( 2x - 5 \) equal to 0 and exclude these. This can be done by setting \( 2x - 5 = 0 \) and solving the equation for \( x \).

\[
2x - 5 = 0
\]

\[
2x = 5
\]

\[
x = \frac{5}{2}
\]

Do not consider \( x = \frac{5}{2} \) when considering the rational expression \( \frac{x+3}{2x-5} \). This expression is defined for all real numbers except \( x = \frac{5}{2} \). Sometimes to shorten the answer it is written as \( x \neq \frac{5}{2} \).

b) To determine the value or values that are excluded, set the denominator equal to zero and solve the equation for the variable.

\[
x^2 + 6x - 7 = 0
\]

\[
(x + 7)(x - 1) = 0
\]

\[
x + 7 = 0 \quad \text{or} \quad x - 1 = 0
\]

\[
x = -7 \quad \text{or} \quad x = 1
\]

Therefore, do not consider the values \( x = -7 \) or \( x = 1 \) when considering the rational expression \( \frac{x+3}{x^2+6x-7} \). Both \( x = -7 \) and \( x = 1 \) make the denominator zero. This is defined for all real numbers except \( x = -7 \) and \( x = 1 \). Thus, \( x \neq -7 \) and \( x \neq 1 \).

**SIGNS OF A FRACTION**

\[
\frac{-a}{b} = \frac{a}{b} = \frac{a}{-b}
\]

Notice: \( \frac{a}{b} \neq \frac{-a}{-b} \)

Generally, a fraction is not written with a negative denominator. For example, the expression \( \frac{-2}{-5} \) would be written as either \( \frac{2}{5} \) or \( \frac{-2}{5} \). The expression \( \frac{x}{-(4-x)} \) can be written \( \frac{x}{x-4} \) since \( -(4-x) = -4 + x \) or \( x - 4 \).
Other examples of equivalent fractions:

\[-\frac{y}{y-2} = -\frac{y}{2-y} \quad \frac{x-2}{x-3} = \frac{2-x}{3-x} \quad \frac{z-5}{7-z} = -\frac{5-z}{z-7}\]

SIMPLIFYING RATIONAL EXPRESSIONS

A rational expression is simplified or reduced to its lowest terms when the numerator and denominator have no common factors other than 1. The fraction \(\frac{9}{12}\) is not simplified because 9 and 12 both contain the common factor 3. When the 3 is factored out, the simplified fraction is \(\frac{3}{4}\).

\[
\frac{9}{12} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{3}{4}
\]

The rational expression \(\frac{ab-b^2}{2b}\) is not simplified because both the numerator and denominator have a common factor, \(b\). To simplify this expression, factor \(b\) from each term in the numerator, then divide it out.

\[
\frac{ab-b^2}{2b} = \frac{b(a-b)}{2b} = \frac{a-b}{2}
\]

Thus, \(\frac{ab-b^2}{2b}\) becomes \(\frac{a-b}{2}\) when simplified.

<table>
<thead>
<tr>
<th>To Simplify Rational Expressions</th>
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<tbody>
<tr>
<td>1. Factor both the numerator and denominator as completely as possible.</td>
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<td>2. Divide out any factors common to both the numerator and denominator.</td>
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**Example 1**

Simplify \(\frac{5x^3+10x^2-25x}{10x^2}\)

Solution  Factor the greatest common factor, \(5x\), from each term in the numerator. Since \(5x\) is a factor common to both the numerator and denominator, divide it out.

\[
\frac{5x^3+10x^2-25x}{10x^2} = \frac{5x(x^2+2x-5)}{5x^2x} = \frac{x^2+2x-5}{2x}
\]

**Example 2**

Simplify \(\frac{x^2+2x-3}{x+3}\)

Solution  Factor the numerator; then divide out the common factor.

\[
\frac{x^2+2x-3}{x+3} = \frac{(x+3)(x-1)}{x+3} = x - 1
\]
Example 3  Simplify $\frac{x^2-16}{x-4}$

Solution  Factor the numerator; then divide out common factors.

\[
\frac{x^2-16}{x-4} = \frac{(x+4)(x-4)}{x-4} = x - 4
\]

Example 4  Simplify $\frac{2x^2+7x+6}{x^2-x-6}$

Solution  Factor both the numerator and denominator, then divide out common factors.

\[
\frac{2x^2+7x+6}{x^2-x-6} = \frac{(2x+3)(x+2)}{(x-3)(x+2)} = \frac{2x+3}{x-3}
\]

Example 5  Simplify $\frac{2m^2+4m-6}{4m^2+16m+12}$

Solution  Factor both the numerator and denominator, then divide out common factors.

\[
\frac{2m^2+4m-6}{4m^2+16m+12} = \frac{2(m+3)(m-1)}{2(m+3)(m+1)} = \frac{m-1}{2(m+1)}
\]

Example 6  Simplify $\frac{x^3-1}{x^2-2x+1}$

Solution  Factor both the numerator and denominator, then divide out common factors.

\[
\frac{x^3-1}{x^2-2x+1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}
\]

Example 7  Simplify $\frac{2x+4-x^3-2x^2}{x^2+5x+6}$

Solution  Factor both numerator and denominator, then divide out common factors.

\[
\frac{2x+4-x^3-2x^2}{x^2+5x+6} = \frac{2(x+2)-x^2(x+2)}{(x+2)(x+3)} = \frac{(x+2)(2-x^2)}{(x+2)(x+3)} = \frac{2-x^2}{x+3}
\]
Consider the expression $\frac{5x-3}{x+3}$, a common student error is to attempt to cancel the $x$ or the 3 or both $x$ and 3 appearing in this expression.

This is WRONG! $\frac{5x-3}{x+3}$ does not equal $\frac{5-1}{1+1} = 2$

It is WRONG because factors are not being reduced. Evaluating this expression for an easy value, such as $x = 1$, would show that the illustrated cancellations are WRONG. If $x = 1$, $\frac{5x-3}{x+3}$ becomes $\frac{5(1)-3}{1+3} = \frac{2}{4} = \frac{1}{2}$.

Remember: Only common factors can be divided out from expressions.

In the denominator of the example on the left, $4x$, the 4 and $x$ are factors since they are multiplied together. The 4 and the $x$ are also both factors of the numerator $20x^2$, since $20x^2$ can be written $4 \cdot x \cdot 5 \cdot x$.

Some students incorrectly divide out terms. In the expression $\frac{x^2-20}{x-4}$, the $x$ and $-4$ are terms of the denominator, not factors, and therefore cannot be divided out.

Recall that when -1 is factored from a polynomial, the sign of each term in the polynomial changes.

EXAMPLES:

$-3x + 5 = -1(3x - 5) = -(3x - 5)$

$6 - 2x = -1(-6 + 2x) = -(2x - 6)$

Example 8

Simplify $\frac{3x-7}{7-3x}$

Solution Since each term in the numerator differs only in sign from its like term in the denominator, factor-1 from each term in the denominator.

$$\frac{3x-7}{7-3x} = \frac{3x-7}{-1(-7+3x)}$$

$$= \frac{3x-7}{-(3x-7)}$$

$$= -1$$

Example 9

Simplify

$$\frac{4x^2-23x-6}{6-x} = \frac{(4x+1)(x-6)}{6-x}$$

$$\frac{(4x+1)(x-6)}{6-x} = \frac{(4x+1)(x-6)}{-1(x-6)}$$

$$= \frac{4x+1}{-1}$$

$$= -(4x + 1)$$
ADDITIONAL EXERCISES

Determine the value or values of the variables for which the expression is defined.

1. \( \frac{x+9}{x^2-x-12} \)
2. \( \frac{x-3}{5x-3} \)
3. \( \frac{x^2+5x-36}{x^2+7x+6} \)
4. \( \frac{64x^2-25}{-8x^2-10x} \)
5. \( \frac{x^2+5x-36}{x^2-8x+12} \)
6. \( \frac{16x^2-9}{-9x^2-10x} \)

Simplify

7. \( \frac{9f+fg}{4f} \)
8. \( \frac{5p+pq}{8p} \)
9. \( \frac{4f+fg}{7f} \)
10. \( \frac{12x-24}{8-4x} \)
11. \( \frac{6x-24}{12-3x} \)
12. \( \frac{x^2+2x-35}{5-x} \)
13. \( \frac{x^2-5x-14}{7-x} \)
14. \( \frac{x^2-5x-14}{x^2-49} \)
15. \( \frac{u-1}{u^2-1} \)
16. \( \frac{x^2-x-12}{x^2-16} \)
17. \( \frac{j+8}{j^2-64} \)
18. \( \frac{x^2+5x-14}{2-x} \)
19. \( \frac{x^2+7x-18}{x^2-4} \)
20. \( \frac{x^2-9}{27-x^4} \)

Answers

1. \( x \neq -3, x \neq 4 \)
2. \( x \neq 3 \)
3. \( x \neq -6, x \neq -1 \)
4. \( x \neq 0, x \neq -\frac{5}{4} \)
5. \( x \neq 6, x \neq 2 \)
6. \( x \neq -\frac{10}{9}, x \neq 0 \)
7. \( \frac{9+g}{4} \)
8. \( \frac{5+g}{8} \)
9. \( \frac{4+g}{7} \)
10. \( -3 \)
11. \( -2 \)
12. \( -(x + 7) \)
13. \( -(x + 2) \)
14. \( \frac{x+2}{x+7} \)
15. \( \frac{1}{u+1} \)
16. \( \frac{x+3}{x+4} \)
17. \( \frac{1}{j-8} \)
18. \( -(x + 7) \)
19. \( \frac{x+9}{x+2} \)
20. \( \frac{-x+3}{9+3x+x^2} \)