Absolute Value Equations and Inequalities

Absolute Value Definition - The absolute value of *x*, is defined as…

$$|\boldsymbol{x}| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$$ where ***x*** is called the "argument"

Steps for Solving *Linear* Absolute Value *Equations*: $i.e.\ \ |ax+b| = c$

1. Isolate the absolute value.

2. Identify what the isolated absolute value is set equal to…
   a. If the absolute value is set **equal to <u>zero</u>**, remove absolute value symbols & solve the equation to get **one solution**.

   b. If the absolute value is set **equal to a <u>negative</u>** number, there is **no solution.**

   c. If the absolute value is set **equal to a <u>positive</u>** number, set the argument (*expression within the absolute value*) equal to the number **and** set it equal to the opposite of the number, using an 'or' statement in between the two equations. Then solve each equation separately to get **two solutions**.

Examples:

a. $|3x+12| + 7 = 7$

$|3x+12| = 0$

Because this equals **0**, there is **ONE** solution.

$3x + 12 = 0$

$3x = -12$

$\boldsymbol{x = -4}$

b. $|3x-7| + 7 = 2$

$|3x-7| = -5$

Because this equals a **negative** number, there is **NO** solution.

No Solution

c. $|3x-7| + 7 = 9$

$|3x-7| = 2$

Because this equals a **positive** number there are **TWO** sltns.

$3x - 7 = 2$ or $3x - 7 = -2$

$3x = 9$ or $3x = 5$

$\boldsymbol{x = 3}$ or $\boldsymbol{x = \frac{5}{3}}$

d. $|x+5| = |2x-1|$ → **Set up <u>two</u> Equations**

$x + 5 = +(2x-1)$ or $x + 5 = -(2x-1)$

$\boldsymbol{x = 6}$ or $x + 5 = -2x + 1 \rightarrow$ $3x = -4 \rightarrow$ $\boldsymbol{x = -\frac{4}{3}}$

Steps for Solving *Linear* Absolute Value *Inequalities*: $i.e \quad |ax+b| \le c$

1. Isolate the absolute value.

2. Identify what the absolute value inequality is set "equal" to…

"Zero"

a. If the absolute value is **less than zero**, there is **no solution.**

b. If the absolute value is **less than or equal to zero**, there is **one solution**. Just set the argument equal to zero and solve.

c. If the absolute value is **greater than or equal to zero**, the solution is **all real numbers.**

d. If the absolute value is **greater than zero**, the solution is all real numbers **except** for the value which makes it equal to zero. This will be written as a **union**.

"Negative" Number

e. If the absolute value is **less than** <u>or</u> **less than or equal to a negative number,** there is **no solution.** The absolute value of something will *never* be less than or equal to a negative number.

f. If the absolute value is **greater than** <u>or</u> **greater than or equal to a negative number,** the solution is **all real numbers.** The absolute value of something will *always* be greater than a negative number.

"Positive" Number

g. If the absolute value is **less than** <u>or</u> **less than or equal to a positive number**, the problem can be approached two ways. Either way, the solution will be written as an **intersection**.

   i. Place the argument in a 3-part inequality (compound) between the opposite of the number and the number, then solve.

   ii. Set the argument less than the number **and** greater than the opposite of the number using an "and" statement in between the two inequalities.

h. If the absolute value is **greater than** <u>or</u> **greater than or equal to a positive number**, set the argument less than the opposite of the number **and** greater than the number using an 'or' statement in between the two inequalities. Then solve each inequality, writing the solution as a **union** of the two solutions.

3. Graph the answer on a number line <u>and</u> write the answer in interval notation.

Examples:

a. $|x-4| \ge 0$

All Real Numbers

b. $|2x-1|+4 < 4$

$|2x-1| < 0$

No Solution

c. $-3+|x+1| \le -3$

$|x+1| \le 0$

Set $x+1=0$

So $\boldsymbol{x=-1}$

d. $|3x + 4| + 5 \leq 3$

$|3x + 4| \leq -2$

No Solution

e. $2|x - 1| - 4 \geq 2$

$2|x - 1| \geq 6$

$|x - 1| \geq 3$

$x - 1 \geq 3 \quad OR \quad x - 1 \leq -3$

$x \geq 4 \quad OR \quad x \leq -2$

$(-\infty, -\mathbf{2}] \cup [\mathbf{4}, \infty)$

f. $|x - 6| + 6 \geq -4$

$|x - 6| \geq -10$

All Real Numbers

g. $|2 - x| < 8$

$2 - x < 8 \quad OR \quad 2 - x > -8$

$-x < -6 \quad OR \quad -x > -10$

$x > 6 \quad OR \quad x < 10$

$(\mathbf{6}, \mathbf{10})$

h. $3|4x - 1| \leq 9$

$|4x - 1| \leq 3$

i. $|x + 6| > 0$

Set $\quad x + 6 \neq 0$

So $\quad x \neq -6$

$(-\infty, -\mathbf{6}) \cup (-\mathbf{6}, \infty)$

Problem "h" can be solved using two different approaches.

Option 1 – Split in to two different Inequalities joined by an "AND" statement (Intersection)

$3|4x - 1| \leq 9$

$|4x - 1| \leq 3$

$4x - 1 \leq 3 \quad AND \quad 4x - 1 \geq -3$

$x \leq 1 \quad AND \quad x \geq -\frac{1}{2}$

$\left[-\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{1}\right]$

Option 2 – Write as a compound inequality (Intersection)

$3|4x - 1| \leq 9$

$|4x - 1| \leq 3$

$-3 \leq 4x - 1 \leq 3$ *(add 1)*

$-2 \leq 4x \leq 4$ *(divide by 4)*

$-\frac{1}{2} \leq x \leq 1$

$\left[-\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{1}\right]$

Steps for Solving *NON- Linear* Absolute Value *Equations*:

Follow the same steps as outlined for the linear absolute value equations, but all answers must be plugged back in to the original equation to verify whether they are valid or not (i.e. ***"Check your answers."***) Occasionally, "extraneous" solutions can be introduced that are not correct and they must be excluded from the final answer.

Examples:

a. $|x^2 + 1| = 5$

2 Equations

$x^2 + 1 = 5$ or $x^2 + 1 = -5$

$x^2 = 4$ or $x^2 = -6$

$\sqrt{x^2} = \sqrt{4}$ or $\sqrt{x^2} = \sqrt{-6}$

$\boldsymbol{x = \pm 2}$ or $x = imaginary!$

Check your answers!

Check: $x = 2$	*Check:* $x = -2$
$\|(2)^2 + 1\| = 5$	$\|(-2)^2 + 1\| = 5$
$\|5\| = 5$	$\|5\| = 5$
$5 = 5$ ✓	$5 = 5$ ✓
$\boldsymbol{x = 2}$ **Works!**	$\boldsymbol{x = -2}$ **Works!**

b. $|x^2 + 5x + 4| = 0$

Only 1 Equation

$x^2 + 5x + 4 = 0$

$(x + 1)(x + 4) = 0$

$x + 1 = 0$ and $x + 4 = 0$

$\boldsymbol{x = -1}$ and $\boldsymbol{x = -4}$

Check your answers!

Check: $x = -1$	Check: $x = -4$
$\|(-1)^2 + 5(-1) + 4\| = 0$	$\|(-4)^2 + 5(-4) + 4\| = 0$
$\|1 - 5 + 4\| = 0$	$\|16 - 20 + 4\| = 0$
$\|0\| = 0 \rightarrow 0 = 0$ ✓	$\|0\| = 0 \rightarrow 0 = 0$ ✓
$\boldsymbol{x = -1}$ **Works!**	$\boldsymbol{x = -4}$ **Works!**

c. $|x + 3| = x^2 - 4x - 3$

2 Equations

$x + 3 = x^2 - 4x - 3$ or $x + 3 = -(x^2 - 4x - 3)$

$x^2 - 5x - 6 = 0$ or $x + 3 = -x^2 + 4x + 3$

$(x - 6)(x + 1) = 0$ or $x^2 - 3x = 0$

$x - 6 = 0$ and $x + 1 = 0$ or $x(x - 3) = 0$

$\boldsymbol{x = 6}$ and $\boldsymbol{x = -1}$ or $\boldsymbol{x = 0}$ and $\boldsymbol{x = 3}$

Check your answers!

Plugging each of the 4 answers into original equation results in …

$x = -1 \rightarrow 2 = 2$ ✓

$x = 6 \rightarrow 9 = 9$ ✓

$x = 0 \rightarrow 3 \neq -3$

$x = 3 \rightarrow 6 \neq -6$

So, the only answers to the problem are $\boldsymbol{x = -1}$ and $\boldsymbol{x = 6}$. *($x = 0$ and $x = 3$ are extraneous).*

Absolute Value Practice Problems

	Problem	Answer	Type
1.	$\|x\| = 8$	$\{-8, 8\}$	Equation, "+" number
2.	$\|x - 2\| = 6$	$\{-4, 8\}$	Equation, "+" number
3.	$\|x + 1\| = 0$	$\{ -1 \}$	Equation, "zero"
4.	$\|x - 4\| = -6$	No Solution	Equation, "-" number
5.	$\|3x + 2\| = 10$	$\{-4, \frac{8}{3}\}$	Equation, "+" number
6.	$\|2x + 5\| + 4 = 3$	No Solution	Equation, "-" number
7.	$2\|4x - 1\| = 6$	$\{-\frac{1}{2}, 1\}$	Equation, "+" number
8.	$\frac{1}{4}\|2x - 6\| + 1 = 2$	$\{1, 5\}$	Equation, "+" number
9.	$-3\|x - 1\| - 6 = 3$	No Solution	Equation, "-" number
10.	$\| x - 7\| + 2 = 2$	$\{ 7 \}$	Equation, "zero"
11.	$\|3x + 2\| = \|x - 6\|$	$\{-4, 1\}$	Equation, | | = | |
12.	$\|x - 4\| = \|4 - x\|$	$\mathbb{R}$	Equation, | | = | |
13.	$\|x\| \leq 2$	$[-2, 2]$	Inequality, "+" nmbr
14.	$\|x + 3\| > 4$	$(-\infty, -7) \cup (-7, \infty)$	Inequality, "+" nmbr
15.	$\|x + 3\| < -6$	No Solution	Inequality, "-" nmbr

16.	$3\|2x-4\| \geq -9$	$\mathbb{R}$	Inequality, "-" nmbr
17.	$2\|x-9\|+6 > 6$	$(-\infty\,,\ 9) \cup (9\,,\infty)$	Inequality, "zero"
18.	$-4\|3x-1\| \geq 8$	No Solution	Inequality, "-" nmbr
19.	$-5\|2x+2\|-3 \geq -3$	$\{-1\}$	Inequality, "zero"
20.	$-10+\frac{1}{2}\|x-4\| \geq -10$	$\mathbb{R}$	Inequality, "zero"
21.	$3\left\|\frac{1}{2}x+2\right\|+6 < 15$	$(-10\,,2)$	Inequality, "+" nmbr
22.	$\|x^2-4\| = -4$	No Solution	Equation, Non-Linear
23.	$\|x^2+9x+14\| = 0$	$x = -2,-7$	Equation, Non-Linear
24.	$\|x^2+1\| = 2x$	$x = 1$	Equation, Non-Linear
25.	$\|x-1\| = x^2+4x-5$	$x = -6,\ 1$	Equation, Non-Linear